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Research Statement

Introduction

The interdisciplinary field of **Inverse Problems** is thriving in exciting and unexpected ways. In an inverse problem, an observation is your data and you aim to determine what set of circumstances led to that data. Such problems are inherently interdisciplinary bringing together scientists from various fields such as mathematics, engineering, physics, biology, chemistry, statistics, astronomy, and medicine, etc. My main area of research involves the use of **nonlinear inversion methods** for an imaging modality called **Electrical Impedance Tomography**, or EIT for short. In EIT, the goal is to reconstruct the conductivity distribution inside of a conductive body by using only current and voltage measurements taken at its boundary. The reconstruction task is a highly *ill-posed* nonlinear inverse problem and requires the use of mathematical techniques from functional and complex analysis, inverse scattering and PDE theory, and numerical analysis. Successfully solving a problem of this kind demands computational as well as theoretical skills, both of which I possess and have demonstrated through my strong research record. Despite the complexities, there are very interesting and accessible problems that are appropriate for both graduate and undergraduate research. Below I give a brief overview of my research interests and projects, plan for future research, and my vision for undergraduate research opportunities.

Research Synopsis

In EIT, a conductive body is probed by harmless electrical currents applied through electrodes placed on its boundary, and the resulting voltages are measured at the electrodes. From these surface measurements, we aim to reconstruct an image of the internal conductivity and/or permittivity of the object. The schematic below in Figure 1 demonstrates the setup; the reconstructions are computed for simulated EIT data using my D-bar algorithm, to be discussed below. The conductivity and permittivity of biological tissues such as heart, lung, blood, and fat are different allowing a medical doctor to then use the EIT images as diagnostic tools. Applications include: monitoring heart and lung function in ICU patients, detection and classification of breast tumors, and brain imaging, as well as non-destructive evaluation, detection of groundwater contamination, oil exploration and land mine detection.

The underlying physical problem, derived from Maxwell's equations, is often called *Calderón's problem*, due to his seminal paper on the inverse conductivity problem [6]. Let $\Omega \subset \mathbb{R}^n$ be a bounded connected set with smooth boundary. For an applied voltage $f \in H^{1/2}(\partial\Omega)$, the voltage potential $u \in H^1(\Omega)$ satisfies the elliptic PDE

$$\nabla \cdot \gamma \nabla u = 0, \quad x \in \Omega \tag{1}$$

with $u|_{\partial\Omega} = f$ and complex-valued admittivity $\gamma(x) = \sigma(x) + i\omega\epsilon(x)$ where σ denotes the conductivity, ϵ the permittivity, and ω the frequency of the current. The boundary voltage and current measurements described above correspond to knowledge of the Dirichlet-to-Neumann (D-N) map $\Lambda_\gamma f = \gamma \frac{\partial u}{\partial \nu}\big|_{\partial\Omega}$ where ν is the outward facing unit normal vector to $\partial\Omega$. The *forward* problem would be to determine the potential u inside Ω for a prescribed boundary voltage f. Alternatively, the *inverse* problem is to recover the coefficient γ in Ω from knowledge of Λ_γ.

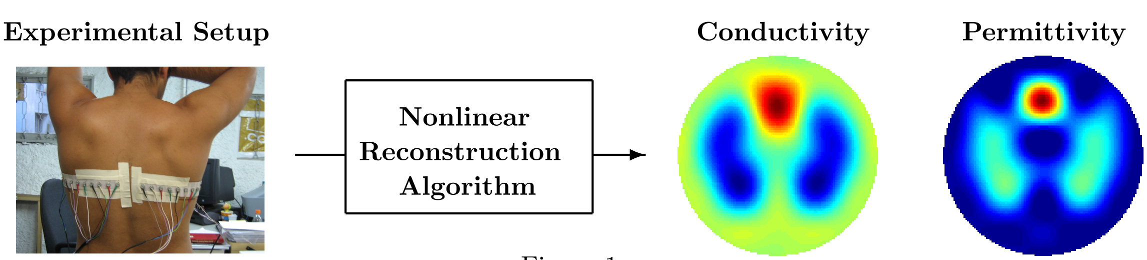


Figure 1

The reconstruction task of EIT, i.e. creating the image, is a nonlinear and highly *ill-posed* inverse problem. It is ill-posed in the sense of Hadamard as small changes in boundary measurements can correspond to large changes in the internal conductivity/permittivity distribution, and in fact has only logarithmic stability making it particularly difficult to solve. The most widely used reconstruction methods are iterative Tikhonov Regularization and linearized approaches. The strong nonlinearity of the EIT problem causes the Tikhonov Regularization method to get stuck in local minima. The linearized approaches are only able to produce difference images, pictures of what has changed with respect to a given time frame (i.e. 1 cardiac cycle, 1 min, 10 min, etc.). Such difference images are useful for monitoring applications but impractical for classification/detection scenarios where a reference image is often unavailable. By using a more mathematically rigorous approach to the inverse problem, one can reconstruct a picture of the conductivity/permittivity at a specified time, i.e. form an absolute image. This ability is particularly important in breast cancer applications as malignant breast tumors are 4-9 times more conductive than benign tumors, and thus the values corresponding to absolute images can play an important role in the classification task.

The ill-posedness and nonlinearity challenges of EIT require the use of special nonlinear inversion methods, of which **D-bar** methods have played a crucial role. D-bar methods are based on tailor-made scattering transforms, nonlinear Fourier transforms, which solve the inverse problem uniquely. They are direct (non-iterative), parallelizable, and solve the full nonlinear EIT problem in a mathematically rigorous and constructive manner using special **Complex Geometrical Optics solutions** (CGOs) which have special growth and decay properties. Two-dimensional D-bar methods consist of the following conceptual ideas [14, 2, 4, 8]:

(i) Transform the physical problem (1) to a Schrödinger/Beltrami equation, or a first-order $\partial\,\overline{\partial}$ system.
(ii) Introduce an auxiliary parameter $k \in \mathbb{C}$ and look for CGO solutions $\big($e.g. asympototic to e^{ikx}, $x \in \mathbb{C}\big)$.
(iii) The CGOs satisfy a $\overline{\partial}_k$ equation which involves a *non-physical scattering transform.*
(iv) The solution to the $\overline{\partial}_k$ equation can be used to recover γ via a low $|k|$ frequency limit of the CGOs.
(v) The scattering data can be computed using the D-N map and boundary traces of the CGOs.

My research both at Colorado State University (CSU) during my PhD, and at the University of Helsinki in the *Centre of Excellence in Inverse Problems Research* (CoE) as a Postdoctoral Researcher, has provided especially significant theoretical and computational breakthroughs in the complex admittivity problem, partial boundary data problem, and anisotropic conductivity problem; each is discussed in more detail below, and has spurred many new projects (see Future Research).

The Complex Admittivity Problem

In my PhD thesis at CSU, under the supervision of Professor Jennifer L. Mueller, I developed the first D-bar algorithm which recovers permittivity in addition to conductivity. Recovering the permittivity gives doctors an extra piece of information making it possible to distinguish between pathologies such as pleural effusion (low permittivity) and atelectasis (high permittivity), both of which have high conductivities. In addition, the permittivity component varies more significantly than the conductivity across a spectrum of applied frequencies, providing more information for Electrical Impedance Spectroscopy (EIS) imaging.

My thesis provided both theoretical and computational developments that turned the constructive existence/uniqueness CGO based proof of Francini [8] into a functional D-bar numerical reconstruction algorithm that works with real world D-N data from electrode measurements. This amounted to the first direct (non-iterative) reconstruction algorithm for admittivities, and resulted in publications in *Inverse Problems* and *Transactions on Medical Imaging*, top journals in the inverse problems and EIT fields. Key contributions from my thesis involve theoretical breakthroughs for points (iv) and (v) above, and the numerical implementation of the resulting algorithm.

The approach is based on transforming the admittivity equation (1) to a first-order $\partial\,\overline{\partial}$ system

$$(D-Q)\Psi = 0, \qquad \text{where } D = \begin{pmatrix} \overline{\partial} & 0 \\ 0 & \partial \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & -\frac{1}{2}\,\partial \log\gamma \\ -\frac{1}{2}\,\overline{\partial}\log\gamma & 0 \end{pmatrix},$$

and looking for solutions $\Psi(x,k)$

$$\left(\begin{array}{cc}\Psi_{11} & \Psi_{12}\\ \Psi_{21} & \Psi_{22}\end{array}\right)=\left(\begin{array}{cc}M_{11} & M_{12}\\ M_{21} & M_{22}\end{array}\right)\left(\begin{array}{cc}e^{ikx} & 0\\ 0 & e^{-ik\bar{x}}\end{array}\right),\qquad \text{where } M\sim\left(\begin{array}{cc}1 & 0\\ 0 & 1\end{array}\right) \text{ as } |x|\to\infty.$$

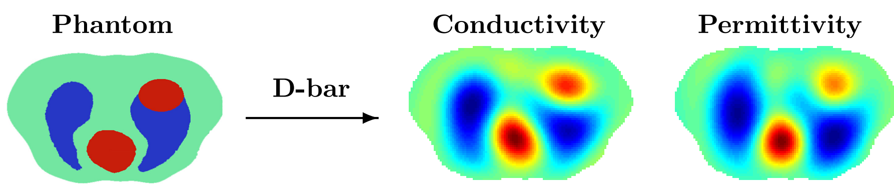
The CGO solutions $M(x,k)$ satisfy a $\overline{\partial}_k$ equation

$$\overline{\partial}_k\, M(x,k)=M\left(x,\bar{k}\right)\left(\begin{array}{cc}e(x,\bar{k}) & 0\\ 0 & e(x,-\bar{k})\end{array}\right)S(k),$$

where $S(k)$ is the nonphysical scattering transform whose diagonal entries are zero, and $e(x,k)=\exp\left(i\left(kx+\bar{k}\bar{x}\right)\right)$ defines a unimodular multiplier.

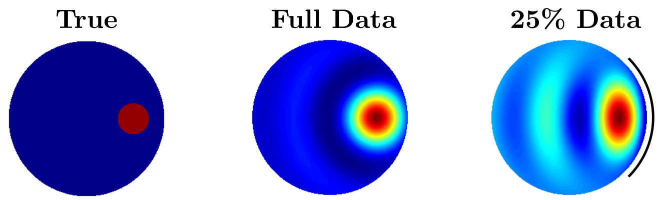
The logarithmic stability of the inverse admittivity problem makes the inversion task extremely challenging even with perfect noise-free data, a real world impossibility, causing the non-physical scattering data to blow up to $\pm\infty$ with k, and making a low $|k|$ frequency limit of the CGOs essential. Francini's proof used a high frequency limit and thus is not a viable option. My result allows the admittivity to be recovered from the CGOs at a single frequency of k, namely $k=0$, thus significantly increasing stability. A key advantage of $\overline{\partial}$ methods is that the scattering data (the $\overline{\partial}_k$ data) is determined by the boundary measurement data. My thesis provides the missing link allowing the scattering data to be determined by solving a Fredholm integral equation of the second kind, with the Faddeev Green's function for the Laplacian as its kernel (as is needed in the well established conductivity only $\overline{\partial}$ methods). The resulting approach is direct, parallelizable, solves the full nonlinear problem, and does not require the solution to the forward problem (i.e. no FEM solution). Furthermore, D-bar approaches in EIT provide a beautiful crossover from inverse scattering theory and integrable systems.

Additionally, I coded the resulting numerical algorithm and it works on simulated FEM data and real world thoracic EIT data. The figure below shows reconstructions from simulated data which used a physically realistic 32 electrodes and modeled the challenges of contact impedance, non-circular domains, and the medically relevant pathology of fluid in a lung. As stated above, these results have led to publications in the field's premier journals [10, 12], with additional manuscripts in the works involving experimental EIT data collected on human patients.



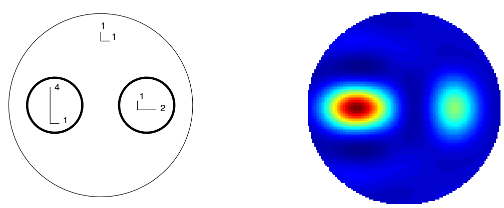
The Partial Boundary Data Problem

Many realistic imaging situations in EIT do not allow for the acquisition of EIT measurements from the entire surface. In applications such as brain and breast imaging, data acquisition takes places on only a proper subset of the boundary such as the top and back of the skull, or the breast itself. The mathematical formulation of the problem requires infinite-precision full boundary knowledge of the D-N map, i.e. the electrical surface measurements. During my time at the CoE in Helsinki, Professor Samuli Siltanen, a pioneer in D-bar methods for EIT and computational methods for inverse problems and imaging, and I have been working to determine the effect of such partial EIT data on existing CGO approaches. We have shown that for low scattering frequencies k, the boundary traces of the CGOs (the common first and most unstable step in the D-bar approaches) can be recovered on the region where the measurements take place. With as little as 25% coverage, existing D-bar methods can be used with the partial D-N information to provide useful information about the internal conductivity/permittivity distributions (see figure below). This project has led to a recent publication in *Contemporary Mathematics* [13], and spurred research along new lines of novel data-driven CGO solution methods (see Future Research) with an additional paper already under review [9].



The Anisotropic Conductivity Problem

Anisotropic tissues are abundant in nature, e.g. the tissue in the human heart is 3 times more conductive in the longitudinal than transverse direction. Often in traditional EIT, only the isotropic problem is considered, thus ignoring such directional preferences. Furthermore, incorrect modeling of the boundary shape corresponds to a change of coordinates giving rise to an anisotropic conductivity problem, even in the case of isotropic tissues. In the 2D anisotropic EIT problem, the scalar valued coefficient γ in (1) is replaced by a 2×2 real valued matrix function σ. It is well known that only the determinant of the coefficient σ can be recovered uniquely [16, 3], and in fact only up to a change of coordinates (a quasi-conformal diffeomorphism), but a stable constructive way to do so has remained a major open question. Recently, together with *Calderón Prize* winning mathematician Professor Matti Lassas, Siltanen and I have developed the first stable constructive proof and reconstruction algorithm for planar anisotropic conductivities. The figure below shows an anisotropic conductivity with two circular inclusions in an isotropic background and its corresponding isotropic reconstruction. The anisotropic preferences squeeze the circular inclusions into ellipses. The proof employs quasi-conformal mapping principles, complex and functional analysis, and scattering theory, with CGOs serving as the backbone. Using simulated anisotropic D-N data, we have tested the approach numerically and demonstrated that pertinent information such as the strength and directional preference of the anisotropy in tissues can be determined from the reconstructions, thus adding a new feature to EIT imaging. The first of three papers coming out of this exciting advancement has recently been submitted for publication [11].



Future Research Plan

During my tenure as a Postdoctoral Researcher I have established a diverse world-wide network of collaborators including leading researchers in inverse problems, integrable systems, seismic imaging, and image processing. Add to this the connections developed during my graduate studies with Professor Mueller, and I am well poised for a promising and productive interdisciplinary research career. In addition to the specific projects and their extensions described above in my Research Synopsis (complex admittivity, anisotropic conductivity, and partial data), the CGO solutions and $\overline{\partial}$ techniques developed during my PhD studies and Postdoctoral work serve as a springboard into a sea of new, exciting, and diverse problems. Below I briefly describe just a small sample of the projects I look forward to investigating, many of which contain aspects suitable for graduate student involvement.

CGO Sinogram

The traces of the CGO solutions obtained in the first step of D-bar approaches for the EIT problem contain more transparent information about the internal conductivity distribution than the voltage measurements or D-N map (see `http://sjhamilton.weebly.com` for an illustrative video). Furthermore, even partial data CGO traces can be recovered for low scattering frequencies k. At the CoE I have been working on a novel approach with Siltanen called a CGO *sinogram*, an analog of the sinogram used in X-ray tomography. The *CGO sinogram* is comprised of a matrix of the CGO boundary traces ranging over a circle $|k| = R$ of fixed radius within the region of proven stability from regularization theory. I plan to use the *CGO sinogram* as a data-driven image enhancement for D-bar reconstructions. A key aspect of the D-bar reconstruction method involves a low-pass filtering (i.e. truncation) of the scattering data when solving the $\overline{\partial}_k$ equation. Such truncation corresponds to rigorous regularization strategy, but inevitably results in smoothed/blurred reconstructions. To combat the loss of information resulting from the low-pass filtering I am interested in investigating the following projects:

(a) **Data-Driven Enhanced D-bar for 2D EIT:** In EIT imaging the approximate ranges of realistic conductivity/permittivity values are often known *a priori*, in addition to the fact that the internal structure often contains sharp jumps/edges (e.g. organ boundaries). Such information is not taken into account in traditional D-bar approaches, and its inclusion can significantly improve reconstructions. Using an edge-preserving process such as *diffusive image segmentation* [1] monitored by the *CGO sinogram*, one can combine regularized 2D D-bar methods with data-driven image processing with contrast enhancement, thus maintaining the rigor of the method and its connection to the data. In collaboration with Siltanen, and a University of Helsinki PhD candidate, our first paper along these lines has recently been submitted [9].

(b) **Data-Driven Enhanced D-bar for 3D EIT:** The approach can be extended to 3D, where a similar regularized D-bar algorithm exists but requires a large scattering frequency limit. With this limit, the scattering data tends towards the Fourier transform. Due to instabilities from noisy D-N data, a low-pass filtering of the data is needed, thus ignoring all of the high frequency Fourier data where sharp features such as edges are known to be. Currently, 3D D-bar methods suffer blurring and a large loss of contrast due to the truncation. By extending the 2D techniques in (a), the aim is to both sharpen the reconstruction, and increase the contrast by using a data-driven image processing D-bar hybrid method. This project will involve collaboration with the Technical University of Denmark and Università di Genova.

(c) **Data-Driven Enhanced D-bar for Partial Data EIT:** The approaches in (a) and (b) may also be extended to the 2D and 3D partial data problems which would be significant breakthroughs for applications such as breast cancer, brain, and thoracic imaging.

Integrable Systems: Davey-Stewartson II

The D-bar methods used in EIT originated from breakthroughs in the integrable systems community for the Novikov-Vesilov (NV) and Davey-Stewartson II (DS-II) evolution equations. Explicit connections between the scattering data of NV and the Schrödinger equation in the conductivity problem have become a recent area of study. Similarly, the systems describing the forward and inverse scattering data of the DS-II system are nearly identical to the first-order $\partial\,\overline{\partial}$ system in my PhD thesis. Along with integrable systems experts Professors Peter Perry (U of Kentucky), Peter Miller (U of Michigan) and Ken McGlaughlin (U of Arizona) I am interested in investigating the existence of exceptional points and solitons for the DS-II system. I have developed the numerical codes to solve the DS-II evolution system for a specified evolution time t with only the initial condition known (i.e. the solution at the previous time steps are not required to produce the solution at the next time step, a key feature of IST approaches). Such a collaboration allows theoretical questions to be quickly tested numerically and researchers to gain insight into new questions to ask. This work is ongoing from the *Exceptional Circles Helsinki Workshop* for which I was a co-organizer this past August (see CV).

Seismic Imaging

The recovery of long wavelengths in the 3D Helmholtz problem in seismic imaging can be related to solving a non-zero energy version of the 3D inverse conductivity problem. In collaboration with expert Professor Maarten de Hoop (Purdue University) and Siltanen, we are working to apply CGO and D-bar techniques which solve the zero-energy (EIT) problem, to the non-zero energy Helmholtz setting. As the data sets are substantially larger for seismic imaging than EIT, and are restricted to partial data, alternative solution methods must be considered and may involve the new *CGO sinogram* as well.

Undergraduate Research

I was first exposed to research in mathematics during my junior year at Saint Michael's College. Before that time, I had no idea what "research" in mathematics meant, and had never considered a career in mathematics outside of teaching. The combination of discovery and creativity was exhilarating and brought with it a new-found interest in research. As part of my REU, I presented my original results [7] at NASA's Goddard Space Station in Maryland, and met with real world researchers. I found this opportunity invaluable in spurring my interest in interdisciplinary research and am committed to offering the same opportunities to my students. The conceptual ideas behind my research interests are straightforward and lend themselves to undergraduate research both with and without a lab setting. Moreover, the multitude of medical and geophysical applications of EIT and inverse problems make my research areas ripe for external funding. Lab ready EIT machines are now available for just over $5,000. Below I describe two of the many undergraduate projects students could readily work on.

Project 1: Regularization Strategies
The most widely used techniques in reconstruction approaches for the EIT problem are regularization strategies. Common methods include Least Squares, Total Variation, and Tikhonov Regularization. Through an independent study or research project, a student comfortable with calculus and linear algebra can learn about one or more of the techniques, use computational software to write their own code, and test the strategies on both simulated and real data. Examples include: sharpening their own photographs (blurred with added noise), X-ray data, and simulated or real EIT data. Armed with basic codes and test problems, students can experiment to study the effect of using various norms (L^2, L^∞, L^1, or a combination thereof), a growing area of much interest in cutting-edge regularization.

Project 2: Leslie/Lefkovitch Matrix Modeling & Sensitivity Analysis
With a basic knowledge of matrix linear algebra and partial derivatives, students have the necessary tools to explore matrix modeling of populations and learn about sensitivity analysis. For a population of the student's choosing, she/he would develop a basic matrix model using an age-structured Leslie matrix model or stage-structured Lefkovitch matrix model (e.g., yearlings, juveniles, adults, post-reproductive adults). Survival, transition, and fecundity parameters can be estimated from the literature, via direct collaboration with departments such as biology, ecology, forestry, etc., or by collaboration with groups such as the National Parks Service. Through such a project, students would gain computational skills and explore how an initial population evolves over time for a given set of parameters. Does the population die out? Grow exponentially? Reach a stable equilibrium? In a hands on approach, the students can vary the parameters themselves to determine how "sensitive" the model is to each. This is the most commonly used approach in ecology. However, with a more mathematical approach, using partial derivatives students can calculate the sensitivity of the model to each parameter explicitly. Management strategies can be studied by introducing harvesting. My experience mentoring interdisciplinary undergraduate teams in FEScUE (see CV) as well as working myself in a diverse interdisciplinary team PrIMES (see CV) have shown me first-hand that such projects are accessible, intriguing, and can lead to peer-reviewed scientific publications [15, 5]; making such a project ripe for undergraduate exploration.

I am committed to contributing to, or establishing, student-led chapters of organizations such as SIAM, AWM, AMS, etc. which provide unique opportunities for undergraduate and graduate students to see the uses of mathematics in industry as well as academia. During my graduate studies, I was a *founding member* of the now thriving CSU chapter of **SIAM**, and served as *Vice President* (see CV). We organized visits and talks from top-notch researchers in various fields of mathematics, took field trips to the National Center for Atmospheric Science (NCAR), and Google in Boulder, CO. The chapter has also provided crash courses in Matlab ranging from basic skills training to powerful parallel computing. The chapter is comprised of undergraduate and graduate mathematics majors and non-majors. As an undergraduate student at SMC, I was *Co-President* of our local **Pi Mu Epsilon** chapter. These experiences have shown me the beneficial effects of such organizations, and I would love to afford others even greater opportunities.

Take Home Message

The interdisciplinary nature and numerous cutting-edge real-world applications of my research result in plentiful external funding opportunities. My future research projects involve both theoretical and computational questions, containing aspects accessible and appropriate for graduate students, as well as undergraduate exploration. Furthermore, through my postdoctoral experience, I have established an extensive network of international collaborators across disciplines. I look forward to working with students and researchers at your school in the near future.

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